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# Instability of unsteady flows or configurations

# Part 1. Instability of a horizontal liquid layer on an oscillating plane

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A layer of viscous liquid with a free surface is set in motion by the lower boundary moving simple-harmonically in its own plane. The stability of this motion is investigated. Since the primary flow is time-dependent, the time variable cannot be separated from at least one space variable, and a new approach must be used to investigate the problem. In this paper the stability of long waves is studied by a perturbation method which has not been applied before to problems of stability of unsteady flows, and it is found that the flow under consideration can be unstable for long waves.

## 1. Introduction

It is well known that problems of stability of unsteady flows are troublesome because their time-dependence precludes the use of the exponential time-factor for the perturbation quantities. Criteria for stability in integral form are not difficult to obtain. But these criteria are not very helpful because they involve the unknown eigenfunctions. One can of course follow the method of Orr (1907), obtain a formula expressing the Reynolds number $R$ as the ratio involving integrals containing eigenfunctions as integrands, and determine a lower bound for the critical Reynolds number by minimizing $R$, allowing all disturbances for the integrands, whether dynamically possible or not, provided only that they satisfy the equation of continuity and the boundary conditions. The non-linearity of the equations of motion is allowed to stand. How far these lower bounds fall short of the mark can be judged by what Orr (1907) obtained for plane Couette flow and plane Poiseuille flow, for both of which he obtained 89. This has also been obtained by Conrad & Criminale (1965), who gave 88·88 for the former and 88·91 for the latter flow. Recently Joseph (1966) showed that Orr's lower bound is wrong for plane Couette flow at least, and gave the even lower number 41·3 as the new reliable lower bound. But it is known that plane Couette flow is stable for all Reynolds numbers and plane Poiseuille flow is stable if an identically defined Reynolds number is below 5250. However, the lower bounds are good for finite disturbances, and even for infinitesimal disturbances such estimates are often useful at least in a transient period, before more significant information is obtained.

Conrad & Criminale (1965) also attempted to generalize Squire's (1933) result for three-dimensional disturbances to apply to unsteady flows. But, as they themselves indicated, the time transformation does not allow the generalization to be a useful one. In fact the stability or instability of a three-dimensional disturbance in a given unsteady two-dimensional flow can be determined by the stability or instability of a two-dimensional disturbance in *another* unsteady two-dimensional flow, which differs from the original one not merely in the Reynolds number, as in the case of steady primary flows, but also in the distribution of the velocity of the primary flow, and which will be different for a different three-dimensional disturbance.

Conrad & Criminale (1965) also attempted to obtain a Rayleigh theorem for time-dependent flows. They assumed the eigenfunction $v$ to have the form

$$v(y,\tau) = \sum_{n=1}^{N} \theta_n(\tau)\,\phi_n(y),$$

in which $\tau$ is a time and $y$ a spatial co-ordinate, obtained a second-order differential equation for each of $\phi_n$, and from it obtained the desired theorem. The conclusion, however, cannot be reached from the equation for *each* $\phi_n$, and must be arrived at by considering the *sum* they assumed. Hence the theorem has not been proved for unsteady flows.

Much of existing work on the stability of unsteady flows is based on the assumption of quasi-steadiness, that is, on the assumption that the stability of an unsteady flow is determined by whether or not it is stable for all the (varying) velocity distributions if each of these distributions is assumed to persist. The most recent study of hydrodynamic stability based on the assumption of quasi-steadiness is that of Currie (1967), in whose paper many other references can be found. It is concerned with the onset of Bénard cells when the primary temperature distribution is time-dependent. If the frequency $\omega_*$ of the primary flow (or temperature) is much less than a reference velocity $V$ divided by a reference length $d$, it can be shown that the approach of quasi-steadiness can predict stability or instability over time intervals small compared with the period $T = 2\pi/\omega_*$. If it predicts instability, the slow variation of the primary flow with time may not materially affect the conclusion. But many flows predicted to be stable by the approach of quasi-steadiness may well turn out to be unstable in the long run. We know at least one instance illustrating the erroneousness of the ordinary approach of quasi-steadiness. Benjamin & Ursell (1954) showed that when a cylinder containing inviscid liquid with a free surface is shaken up and down with a simple-harmonic acceleration of amplitude $a_0$, the fluid can be unstable even if $a_0$ is very much less than the gravitational acceleration $g$, whereas an analysis based on quasi-steadiness predicts stability in that case. This paper presents another illustration, which is believed to be the first one for the instability of a viscous fluid. The method used can be applied to many other problems concerning hydrodynamic stability of time-dependent flows or configurations.

We consider here the stability of a primary flow which is completely unsteady, in the sense that it contains no steady part whatsoever. The approach of quasi-steadiness is discarded, and we seek to investigate the instability of the flow in the long run. The particular flow under investigation is described in the next section.

## 2. Primary flow

A horizontal layer of liquid of depth $d$, viscosity $\mu$, and density $\rho$ is set in motion by the lower boundary moving in the $X$-direction (figure 1) with velocity

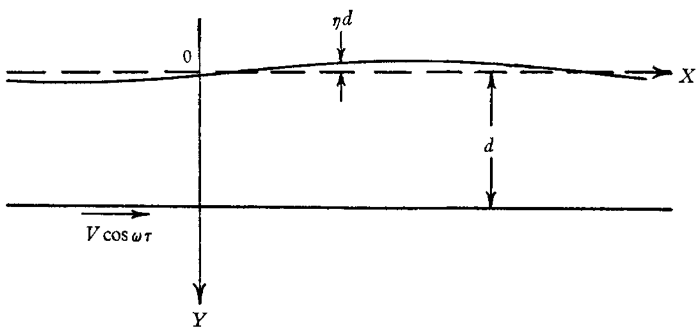


FIGURE 1. Definition sketch

$V \cos \omega_* t$, $\omega_*$ being the frequency, $t$ the time, and $V$ the amplitude of the forcing motion. We shall introduce the dimensionless independent variables

$$\tau = Vt/d, \quad x = X/d, \quad y = Y/d. \tag{1}$$

In terms of these, the equation governing the primary flow is

$$\frac{\partial U}{\partial \tau} = \frac{1}{R}\frac{\partial^2 U}{\partial y^2}, \tag{2}$$

in which $\quad U = \bar{u}(y, \tau)/V, \quad R = Vd/\nu = \text{the Reynolds number}, \quad (3)$

$\bar{u}$ being the velocity of the primary flow and $\nu$ the kinematic viscosity. The boundary conditions for $U$ are

$$\partial U/\partial y = 0 \quad \text{at} \quad y = 0, \tag{4}$$

and $\quad U = \cos \omega_* t = \cos \omega\tau \quad \text{at} \quad y = 1, \quad \text{with} \quad \omega = \omega_* d/V. \quad (5)$

The solution of (2), (4) and (5) is

$$U = A[W + W^* - i \tanh \beta \tan \beta (W - W^*)], \tag{6}$$

with the asterisk as a superscript indicating complex conjugate, and

$$W = \cosh[\beta(1+i)y]\, e^{i\omega\tau}, \quad A = \frac{\cos\beta \cosh\beta}{2(\cos^2\beta + \sinh^2\beta)}, \quad \beta = \left(\frac{\omega R}{2}\right)^{\frac{1}{2}}.$$

The $U$ given by (6) is real, as is obvious by inspection. It is written in that form in order to facilitate the integration to be performed later.

The pressure $\bar{p}$ for the primary flow is hydrostatic, and is zero at $y = 0$. Hence

$$\bar{p} = g\rho Y = g\rho d y. \tag{7}$$

## 3. Formulation of the stability problem

Although Squire's theorem cannot be generalized in a useful way to justify the consideration of two-dimensional disturbances only, it is still true that the stability of a three-dimensional disturbance can be determined from that of a two-dimensional one for a different flow. Hence the method of solution for two-dimensional disturbances will apply to three-dimensional ones. We shall consider only two-dimensional disturbances in this paper. If $u$ and $v$ denote the velocity components in the directions of increasing $x$ and $y$, respectively, and $p$ denotes the pressure, and if

$$u_1 = u/V, \quad v_1 = v/V, \quad p_1 = p/\rho V^2,$$

the equations of motion can be written as

$$\frac{\partial u_1}{\partial \tau} + u_1 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_1}{\partial y} = -\frac{\partial p_1}{\partial x} + \frac{1}{R} \Delta u_1, \tag{8}$$

$$\frac{\partial v_1}{\partial \tau} + u_1 \frac{\partial v_1}{\partial x} + v_1 \frac{\partial v_1}{\partial y} = -\frac{\partial p_1}{\partial y} + F^{-2} + \frac{1}{R} \Delta v_1, \tag{9}$$

in which $\Delta$ denotes the Laplacian operator in $x$ and $y$, and $F$ denotes the Froude number defined by

$$F = V(gd)^{-\frac{1}{2}}. \tag{10}$$

The equation of continuity is

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0. \tag{11}$$

Resolving each of the dependent variables into a primary part and a perturbation part, we have

$$u_1 = U + u', \quad v_1 = v', \quad p_1 = P + p', \tag{12}$$

in which $P$ is $\bar{p}/\rho V^2$, and the accented quantities are the perturbation quantities. If (12) is substituted into (8) and (9), the terms pertaining to the primary flow only, being in balance, are subtracted out, and quadratic terms in the perturbation quantities are neglected, the resulting equations are

$$u'_\tau + U u'_x + U_y v' = -p'_x + \frac{1}{R} \Delta u', \tag{13}$$

$$v'_\tau + U v'_x = -p_y + \frac{1}{R} \Delta v', \tag{14}$$

with the subscripts indicating partial differentiation. Equation (11) can be replaced by

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0,$$

which permits us to use a streamfunction $\psi$ and to write

$$u' = \psi_y, \quad v' = -\psi_x. \tag{15}$$

With (15), the equations of motion can be written as

$$\psi_{y\tau} + U \psi_{xy} - U_y \psi_x = -p'_x + \frac{1}{R} \Delta \psi_y, \tag{16}$$

$$\psi_{x\tau} + U \psi_{xx} = p'_y + \frac{1}{R} \Delta \psi_x. \tag{17}$$

Since $U$ depends on $\tau$ as well as $y$, (16) and (17) do not permit the use of the exponential time-factor. However, as far as $x$ is concerned, we can still consider any disturbance to be a Fourier integral of disturbances simply periodic in $x$, and (16) and (17) allow us to write

$$\psi = \phi(y,\tau)\,e^{i\alpha x}, \quad p' = f(y,\tau)\,e^{i\alpha x}, \tag{18}$$

and (16) and (17) as

$$\phi_{y\tau} + i\alpha U\phi_y - i\alpha U_y\phi = -i\alpha f + \frac{1}{R}\,(\phi_{yyy} - \alpha^2\phi_y), \tag{19}$$

$$i\alpha\phi_\tau - \alpha^2 U\phi = f_y + \frac{i\alpha}{R}(\phi_{yy} - \alpha^2\phi). \tag{20}$$

Elimination of $f$ from (19) and (20) produces

$$R\left[\left(\frac{\partial}{\partial\tau} + i\alpha U\right)(\phi'' - \alpha^2\phi) - i\alpha U_{yy}\phi\right] = \phi'''' - 2\alpha^2\phi'' + \alpha^4\phi, \tag{21}$$

in which the primes on $\phi$ indicate differentiations with respect to $y$.

The boundary conditions at the bottom are the non-slip conditions

$$\text{(i)}\quad \phi(1,\tau) = 0, \quad \text{(ii)}\quad \phi'(1,\tau) = 0. \tag{22}$$

The boundary conditions at the free surface are that both the shear stress and the normal stress there must be zero. Since the shear stress of the primary flow and $\bar{p}$ both vary with $y$, and the free surface will not be flat, the free-surface boundary conditions must include contributions from the primary flow as a result of the vertical displacement of the free surface. If this displacement is denoted by $\eta d$, the relationship between $\eta$ and $\psi$ is provided by the kinematic condition

$$\left[\frac{\partial}{\partial\tau} + U(0,\tau)\frac{\partial}{\partial x}\right]\eta = v' = -\psi_x. \tag{23}$$

With

$$\eta = h(\tau)\,e^{i\alpha x},$$

(23) becomes

$$\left[\frac{d}{d\tau} + i\alpha U(0,\tau)\right]h = -i\alpha\phi(0,\tau). \tag{24}$$

The condition of zero shear at the free surface is

$$U_{yy}(0,\tau)\,\eta + \psi_{yy} - \psi_{xx} = 0,$$

or (iii)

$$U_{yy}(0,\tau)\,h + \phi''(0,\tau) + \alpha^2\phi(0,\tau) = 0. \tag{25}$$

The condition that the normal stress vanish at the free surface is, in its primitive form,

$$\left(-p_1 + \frac{2}{R}\frac{\partial v_1}{\partial y}\right)\rho V^2 + T\,\frac{\partial^2(\eta d)}{\partial X^2} = 0, \tag{26}$$

in which $T$ is the surface tension. Since, by (7),

$$P = \frac{gd}{V^2}y = F^{-2}y,$$

(26) can be written as

$$-F^{-2}\eta - p' - \frac{2}{R}\psi_{xy} + S\eta_{xx} = 0, \tag{27}$$

in which

$$S = T/\rho d V^2.$$

Using (19) to evaluate $f$ and then (18) to evaluate $p'$, and substituting the result into (27), we have (remembering $U_y = 0$ at $y = 0$)

$$i\alpha(F^{-2}+S\alpha^2)\eta+\frac{1}{R}(\psi_{yyy}-3\alpha^2\psi_y)-\psi_{y\tau}-i\alpha U\psi_y = 0,$$

or (iv) $$i\alpha(F^{-2}+S\alpha^2)h+\frac{1}{R}(\phi'''-3\alpha^2\phi')-\phi'_\tau-i\alpha U\phi' = 0, \tag{28}$$

to be applied at $y = 0$.

The differential system governing stability consists of (21), (22), (24), (25) and (28).

## 4. A discussion of the approach of quasi-steadiness

Since the approach of quasi-steadiness is so often used, a discussion of its usefulness and its limitations is desirable. Consider the Orr–Sommerfeld equation (21), to be solved with linear homogeneous boundary conditions. The $U$ in (21) is assumed to be a function of $y$ with at least some of its coefficients containing $\cos\omega\tau$ or $\sin\omega\tau$. For clarity we shall denote $\omega\tau$ by $\tau'$. The following expansion is assumed for $\phi$:

$$\phi = e^{\sigma\tau}\{\phi_0(y,\tau')+\omega\phi_1(y,\tau')+\omega^2\phi_2(y,\tau')+\ldots\}, \tag{29}$$

with $$\sigma = \sigma_0+\omega\sigma_1+\omega^2\sigma_2+\ldots. \tag{30}$$

Expansions (29) and (30) are then substituted into (21) and the boundary conditions, and terms of equal power in $\omega$ are sorted out. The first two equations so obtained are

$$L\phi_0 \equiv \phi_0''''-2\alpha^2\phi_0''+\alpha^4\phi_0 = R[(\sigma_0+i\alpha U)(\phi_0''-\alpha^2\phi_0)-i\alpha U''\phi_0] = 0, \tag{31}$$

$$L\phi_1 = R\sigma_1(\phi_0''-\alpha^2\phi_0)+R\frac{\partial}{\partial\tau'}(\phi_0''-\alpha^2\phi_0), \tag{32}$$

and the other equations are similar in form. In the same way the boundary conditions can be expressed in terms of $\phi_0$, $\phi_1$, etc. In (31) and (32) all accents mean differentiation with respect to $y$. It is immediately evident that the differential system (though we have not written out the boundary conditions explicitly) governing $\phi_0$ is just that used if the approach of quasi-steadiness is used. Note that $\sigma_0$ and $\phi_0$ are determined with the $\tau'$ in $U$ and $U''$ serving only as a parameter. Hence $\sigma_0$ is a function of $\tau'$ and $\phi_0$ contains $\tau'$. After $\sigma_0$ and $\phi_0$ are determined, (32) and the pertinent boundary conditions will determine $\sigma_1$ and $\phi_1$. It is very important in this connexion to observe that: (*a*) the *homogeneous parts* of the differential equations (31), (32), etc., are identical in form; (*b*) the homogeneous parts of the boundary conditions in successive stages of determination are also identical in form; (*c*) and therefore $\sigma_0$ can be considered once for all the eigenvalue pertaining to the operator $L$ and the linear operators in the homogeneous parts of the boundary conditions. It is the item (*c*) that allows us to determine $\sigma_1$, $\sigma_2$, etc. Otherwise since (32) and the boundary conditions for $\phi_1$ are non-homogeneous any $\sigma_1$ would do.

Presumably the expansions (29) and (30) are convergent for any finite $\omega$. If $\omega$ is very small, the quasi-steadiness approach gives good results. For an $\omega$ not very small, more terms in (31) and (32) may have to be taken. Obviously if

$\omega$ is very much greater than 1 the quasi-steadiness approach cannot be relied upon to give good results.

In any practical application of the approach just outlined, the success would depend on the ease with which the particular solution of (32) for $\phi_1$ is obtained. We see at once that, $U$ being dependent on $y$, the task of solving for (31) is already in general rather difficult, let alone the task of finding particular solutions for (32) and the subsequent equations. However, for long waves (31) and (32) can be solved by a power expansion in $\alpha$, and, provided the boundary conditions allow a determination of $\sigma_0$ which does not correspond to rapidly damped disturbances, solutions indicating long-term instability are quite readily obtainable. In fact, if a free surface or an interface exists, the boundary conditions are such that such solutions are indeed obtainable. This paper illustrates in detail how the stability of an unsteady flow with a free surface can be studied by an expansion in $\alpha$. It turns out that the expansion in $\omega$ is then unnecessary.

## 5. Extension of the Floquet theorem

Before we present the solution to the problem formulated in § 3, we shall present an extension of the famous Floquet theorem to the realm of partial differential equations. Since this extension must have wide applications to mathematical physics, we shall consider a more general equation than the Orr–Sommerfeld equation, and general linear boundary conditions. What we seek to establish is the

*Theorem.* Given the differential equation

$$\frac{\partial}{\partial \tau}\sum_{i=0}^{m-1} f_i(y)\, D^i\phi = D^m\phi + \sum_{i=0}^{m-1} g_i(y,\tau)\, D^i\phi \tag{33}$$

in the domain $\qquad 0 \leqslant y \leqslant 1, \quad 0 \leqslant t < \infty,$

in which $f_i(y)$ and $g_i(y,\tau)$ can be expanded into a power series in $y$ in the domain, $g_i(y,\tau)$ is periodic in $\tau$ with period $T$, and $D^i\phi$ is the $i$-th derivative of $\phi$ with respect to $y$, and given the boundary conditions

$$\frac{\partial}{\partial \tau}\sum_{i=0}^{m-1} h_{ji} D^i\phi + \sum_{i=0}^{m-1} k_{ji} D^i\phi = 0 \quad \text{at} \quad y = 0, \tag{34}$$

$$\frac{\partial}{\partial \tau}\sum_{i=0}^{m-1} p_{ji} D^i\phi + \sum_{i=0}^{m-1} q_{ji} D^i\phi = 0 \quad \text{at} \quad y = 1, \tag{35}$$

the independent solutions can be written in the form

$$e^{\mu_1\tau}\chi(y,\tau), \tag{36}$$

in which $\chi(y,\tau)$ is either a periodic function of $\tau$ with period $T$ or can involve polynomials in $\tau$ as well as periodic functions in $\tau$ with period $T$. The $h$ and $p$ in (34) and (35) are constants, the $k$ and $q$ can be periodic functions of $\tau$ with period $T$.

The demonstration will be sketched out briefly here. The function $\phi$ can be expanded in the form

$$\phi = \phi_0(\tau) + \phi_1(\tau)y + \phi_2(\tau)\,y^2 + \ldots + \phi_{n-1}(\tau)\,y^{n-1} + \ldots. \tag{37}$$

If (37) is substituted into (33), (34) and (35), and equal powers in $y$ are collected in the case of (33), and if we stop at the term $\phi_{n-1}(\tau)y^{n-1}$, we have a total of $n$ first-order ordinary differential equations in $\tau$, involving $n$ unknowns

$$\phi_i \quad (i = 0, \ldots, n-1).$$

These can be reduced to one $n$-th order differential equation with the single unknown $\phi_k$, say. Then Floquet's theorem (Ince 1944, p. 381; or Coddington & Levinson 1955, pp. 78–81) states that $\phi_k$ is of the form

$$e^{\mu_1\tau}P(\tau), \tag{38}$$

in which $P(\tau)$ is periodic with period $T$, if the characteristic root $\mu_1$ is simple, or of the form

$$e^{\mu_1\tau}P(\tau) \times \text{a polynomial in } \tau, \tag{39}$$

if $\mu_1$ is a multiple root of a secular equation. Now all the other $\phi_i$ $(i \neq k)$ can be expressed as a linear form of $D^i\phi_k (i = 0, \ldots, n-2)$, by an elimination procedure applied to the $n$ first-order equations. Hence the factor $e^{\mu_1\tau}$ is in all the $\phi_i$. If $\mu_1$ is simple, $\phi_0, \phi_1, \ldots$, and $\phi_{n-1}$ are all of the form (38). Otherwise they are of the form (39), with the same $\mu_1$. Thus the theorem is true up to $n$-terms. As the number of terms taken in (37) is increased, more and more characteristic roots appear, corresponding to more and more modes. Presumably the series (37) will converge to the solution wanted, and there will be a discretely infinite number of modes. Aside from the question of convergence, which the author must leave to more capable minds, the theorem is established.

The theorem can also be established in the original manner of Floquet, if we assume that there exist a complete set of discretely infinite number of solutions

$$\phi_1(y,\tau), \quad \phi_2(y,r), \ldots.$$

Since $\phi_i(y,\tau+T)$ is obviously also a solution, we have

$$\phi_i(y,\tau+T) = \sum_{j=1}^{\infty} a_{ij}\phi_j(y,\tau). \tag{40}$$

We seek a solution $w_i(y,\tau)$ such that

$$w_i(y,\tau+T) = sw_i(y,\tau). \tag{41}$$

Now since the set $\phi_i(y,\tau)$ is complete,

$$w_i(y,\tau) = \sum_{j=1}^{\infty} b_{ij}\phi_j(y,\tau). \tag{42}$$

Substituting (40) and (42) into (41), and demanding that not all the $b$'s be zero. we have

$$\begin{vmatrix} a_{11}-s & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22}-s & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33}-s & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0. \tag{43}$$

If $\mu_1$ is a simple root of (43), then writing

$$s = e^{\mu_1 T},$$

(41) can be written as

$$\frac{w_1(y,\tau+T)}{e^{\mu_1(\tau+T)}} = \frac{w_1(y,\tau)}{e^{\mu_1\tau}},$$

which is to say that
$$w_1(y,\tau)\,e^{-\mu_1\tau}$$
is periodic. Hence the form (36) follows. The case of multiple roots has been discussed in the preceding paragraph, and we shall not dwell upon the subject any longer, except to say that the convergence of any particular root of (43) to a definite value as the the number of rows (and columns) is increased has been left unproven. The arguments advanced in this section would be complete if the convergence of (37) had been established. But in their present form they are so highly plausible that we need have no doubt of the truth of the theorem.

## 6. Solution of the problem

We shall now apply the results of § 5 to the solution of the problem. Assuming $\mu_1$ to be a simple root of (43), we shall write
$$\phi(\tau,y) = e^{\mu_1\tau}\chi(\tau,y),$$
$$h(\tau) = e^{\mu_1\tau}H(\tau),$$
in which $\chi(\tau,y)$ and $H(\tau,y)$ are periodic in $\tau$.

Since we expect to find instability for long waves, or small wave numbers, we shall follow the approach used in Yih (1963) and write
$$\left.\begin{aligned}\chi(\tau,y) &= \phi_0(y,\tau)+\alpha\phi_1(y,\tau)+\alpha^2\phi_2(y,\tau)+\ldots,\\ H(\tau) &= h_0(\tau)+\alpha h_1(\tau)+\alpha^2h_2(\tau)+\ldots,\\ \mu_1 &= \theta_0+\alpha\theta_1+\alpha^2\theta_2+\ldots,\end{aligned}\right\} \tag{44}$$
in which the $\phi$'s and $h$'s are all periodic in $\tau$, and the $\theta$'s are *real* constants. Collecting terms of equal powers in $\alpha$ in equations (21) and (24), we obtain a series of equations in $y$ and $\tau$. The first of the series constituting (24) is
$$\frac{\partial h_0}{\partial\tau}+\theta_0h_0 = 0.$$
Since $h_0$ must be periodic in $\tau$, it follows immediately that†
$$\theta_0 = 0,$$
and without loss of generality $\qquad h_0 = 1. \qquad (45)$

Then for the first approximation the differential system is
$$R\frac{\partial}{\partial\tau}\phi_0'' = \phi_0'''', \tag{21a}$$
(i$a$) $\phi_0(1,\tau) = 0$, (ii$a$) $\phi_0'(1,\tau)=0$,

(iii$a$) $U''(0,\tau)+\phi_0''(0,\tau) = 0$, (iv$a$) $\phi_0'''-R(\phi_0')_\tau = 0$ at $y = 0$.

The solution of the system is simply
$$\phi_0 = -U(y,\tau)+B_0(\tau)+D(V_0+V_0^*)+iE(V_0-V_0^*), \tag{46}$$
in which $\qquad V_0 = \sinh[\beta(1+i)y]\,e^{i\omega\tau},\quad B_0 = b_0\,e^{i\omega\tau}+\bar{b}_0\,e^{-i\omega\tau},$
$$b_0 = \frac{A(1-i\gamma)}{\cosh\beta(1+i)},\quad \gamma = \tanh\beta\tan\beta,$$
$$D+iE = A(1-i\gamma)\tanh\beta(1+i),$$
$D$ and $E$ being real numbers. Note that the $\tau$-dependence of $\phi_0$ is dictated by (iii$a$).

† The other possibility is $h_0 = 0$, $\theta_0 \neq 0$. But it can and will be shown that this leads to a damped mode as far as long waves are concerned. Imaginary values for $\theta_0$, with $h_0 \neq 0$, lead to nothing new.

For the next approximation, (24) gives

$$\frac{dh_1}{d\tau}+\theta_1 = -iB_0(\tau).$$

Since $B_0$ contains only $\sin\omega\tau$ and $\cos\omega\tau$, and $h_1$ is purely periodic,

$$\theta_1 = 0$$

and $$h_1 = -i\int B_0(\tau)\,d\tau = -\frac{1}{\omega}(b_0\,e^{i\omega\tau}-b_0^*\,e^{-i\omega\tau}), \tag{47}$$

the constant of integration being chosen to be zero so that the term independent of $\tau$ in $h$ is $h_0(=1)$ once and for all. This practice will not affect the criterion of stability in the least, and will be followed in calculating $h_2$. Equation (32) shows that even at the second approximation no instability is manifested.

For $\phi_1$, the governing system is, since $\theta_1 = 0$,

$$iR(U\phi_0''-U''\phi_0) = \phi_1''''-R\frac{\partial}{\partial\tau}\phi_1'', \tag{21b}$$

$$\text{(i}b\text{)}\quad \phi_1(1,\tau) = 0, \qquad \text{(ii}b\text{)}\quad \phi_1'(1,\tau) = 0,$$

$$\text{(iii}b\text{)}\quad U''(0,\tau)\,h_1+\phi_1''(0,\tau) = 0,$$

$$\text{(iv}b\text{)}\quad iF^{-2}+\frac{1}{R}\phi_1'''-\frac{\partial}{\partial\tau}\phi_1'-iU\phi_0' = 0 \text{ at } y=0.$$

The $\tau$-dependence of $\phi_1$ is dictated by this system. Instead of terms containing $e^{i\omega\tau}$ and $e^{-i\omega\tau}$, those containing $e^{\pm i2\omega\tau}$ or no $\tau$ at all must be used. The particular solution of (21*b*) is

$$\phi_{1p} = iR\left\{\frac{1}{2}\int\int\left[\left(\phi_0'\int U\right)-U'\int(\phi_0-B_0)\right]-J\right\}, \tag{48}$$

in which $J$ satisfies the equation

$$J''''-R\frac{\partial}{\partial\tau}J'' = B_0\,U'', \tag{49}$$

and the integrations are with respect to $y$, with $dy$ omitted in each integration. After some straightforward integrations, we obtain

$$\phi_{1p} = iR(I_0+I_1+I_2+I_0^*+I_1^*+I_2^*), \tag{50}$$

in which $$I_0 = \frac{iA^2(1+\gamma^2)}{4\omega R}\tanh\beta(1+i)(\sinh 2\beta y+i\sin 2\beta y), \tag{51}$$

$$I_1 = \frac{iA^2(1+\gamma^2)}{\omega R\cosh\beta(1-i)}\cosh[\beta(1+i)\,y], \tag{52}$$

$$I_2 = -\frac{iA^2(1-i\gamma)^2}{\omega R\cosh\beta(1+i)}\cosh[\beta(1+i)\,y]\,e^{i2\omega\tau}. \tag{53}$$

The complementary solution is of the form

$$\phi_{1c} = A_1+B_1y+C_1y^2+D_1y^3+E_1V_1+F_1W_1+E_1^*V_1^*+F_1^*W_1^*+G(\tau)+K(\tau)\,y, \tag{54}$$

in which the eight coefficients are constants, and

$$V_1 = \sinh[\beta(1+i)\,y]\,e^{i2\omega\tau}, \quad W_1 = \cosh[\beta(1+i)\,y]\,e^{i2\omega\tau},$$

$$G(\tau) = g_1\,e^{i2\omega\tau}+g_1^*\,e^{-i2\omega\tau}, \quad K(\tau) = k_1\,e^{i2\omega\tau}+k_1^*\,e^{-i2\omega\tau}.$$

Now a look at (24) reveals that, for prediction of stability for long waves, none but the time-independent terms in

$$\phi_1 = \phi_{1p} + \phi_{1c} \tag{55}$$

and in $U(0,\tau)\,h_1$ need be calculated. For higher approximations at larger $\alpha$ we need to include the terms containing $I_2, E_1, F_1$, $G(\tau)$ and $K(\tau)$, but not at this stage. Considering then the terms independent of $\tau$, we have, in the place of $\phi_1$, the function

$$\Phi(y) = iR(I_0 + I_0^* + I_1 + I_1^*) + A_1 + B_1 y + C_1 y^2 + D_1 y^3, \tag{56}$$

satisfying the conditions

(i$c$) $\Phi(1) = 0$, (ii$c$) $\Phi'(1) = 0$,

(iii$c$) $\Phi''(0) + iA^2(1+\gamma^2)\,R[\operatorname{sech}\beta(1+i) + \operatorname{sech}\beta(1-i)] = 0$,

(iv$c$) $iF^{-2} + (1/R)\Phi''' - iA^2\beta(1+\gamma^2)[(1+i)\tanh\beta(1+i) + (1-i)\tanh\beta(1-i)] = 0.$

The solution is given by the equations

$$\left.\begin{aligned} C_1 &= 0, \quad D_1 = -\frac{i}{6}RF^{-2}, \\ A_1 &= 2D_1 - \frac{i2A^2(1+\gamma^2)}{\omega}(R_1 - R_2), \\ B_1 &= -3D_1 - \frac{iRA^2(1+\gamma^2)}{\beta}R_2, \end{aligned}\right\} \tag{57}$$

in which $R_1$ and $R_2$ are respectively the real part of

$$\frac{i}{4}\tanh\beta(1+i)(\sinh 2\beta + i\sin 2\beta) + \frac{i}{\cosh\beta(1-i)}\cosh\beta(1+i),$$

$$\frac{i\beta}{2}\tanh\beta(1+i)(\cosh 2\beta + i\cos 2\beta) - \frac{2\beta}{(1+i)\cosh\beta(1-i)}\sinh\beta(1+i).$$

Now, according to (24),

$$\frac{dh_2}{d\tau} = -\theta_2 - i[U(0,\tau)\,h_1 + \phi_1(0,\tau)]. \tag{58}$$

The term in $U(0,\tau)\,h_1$ which is independent of $\tau$ is

$$-\frac{i2A^2(1+\gamma^2)}{\omega}I_3, \tag{59}$$

in which $I_3$ is the coefficient of the imaginary part of sech $\beta(1+i)$. The part of $\phi_1(0,\tau)$ which is independent of $\tau$ is

$$\Phi(0) = A_1 + \frac{i2A^2(1+\gamma^2)}{\omega}I_3. \tag{60}$$

Thus the part on the right-hand side of (58) which is independent of time is simply $-\theta_2 - iA_1$, and this must be zero since $h_2$ is periodic in $\tau$. Hence

$$\theta_2 = -iA_1,$$

and the criterion sought is that the flow is unstable or stable according as $-iA_1$

is positive or negative, or according as

$$\frac{6A^2(1+\gamma^2)}{\omega R}(R_2-R_1) > \text{ or } < F^{-2}. \tag{61}$$

| $\beta$ | $L\times 10^n$ | $n$ | $\beta$ | $L\times 10^n$ | $n$ |
|---|---|---|---|---|---|
| 0·10 | 8·0042 | 5 | 3·20 | −2·4914 | 3 |
| 0·20 | 1·2772 | 3 | 3·40 | −1·7888 | 3 |
| 0·30 | 6·4093 | 3 | 3·60 | −1·1007 | 3 |
| 0·40 | 1·9785 | 2 | 3·80 | −5·7129 | 4 |
| 0·50 | 4·6005 | 2 | 4·00 | −2·2517 | 4 |
| 0·60 | 8·7539 | 2 | 4·20 | −3·1780 | 5 |
| 0·80 | 1·9975 | 1 | 4·40 | 5·5388 | 5 |
| 1·00 | 2·7859 | 1 | 4·60 | 7·9145 | 5 |
| 1·20 | 2·7075 | 1 | 4·80 | 7·1305 | 5 |
| 1·40 | 2·0792 | 1 | 5·00 | 5·2075 | 5 |
| 1·60 | 1·3790 | 1 | 5·20 | 3·2385 | 5 |
| 1·80 | 8·2447 | 2 | 5·40 | 1·6890 | 5 |
| 2·00 | 4·4619 | 2 | 5·60 | 6·6071 | 6 |
| 2·20 | 2·1067 | 2 | 5·80 | 8·0065 | 7 |
| 2·40 | 7·5952 | 3 | 6·00 | −1·8288 | 6 |
| 2·60 | 7·1058 | 4 | 7·00 | −5·2612 | 7 |
| 2·80 | −2·1722 | 3 | 8·00 | −7·6320 | 8 |
| 3·00 | −2·8502 | 3 | 9·00 | 6·8866 | 10 |
| | | | 10·00 | −1·6199 | 9 |

TABLE 1

Table 1 shows the variation of the left-hand side ($L$) of (61) with $\beta$. The maximum value of $L$ is 0·279. When $F^{-2}$ is greater than this value, there is stability. Table 1 is shown graphically in figure 2. If $L$ is positive, there can be instability even for small Reynolds numbers provided $\omega_*$ and $F$ are sufficiently large. It is interesting to see that for certain ranges of $\beta$ the value of $L$ is negative, so that in those ranges the motion actually stabilizes the free surface against the formation of long waves. For very small frequencies such that $\omega R = 2\beta^2 \ll 1$, the criterion derived from (61) is that the flow is unstable or stable according as

$$\tfrac{4}{5}A^2\omega^2R^2 > \text{ or } < F^{-2}, \tag{61a}$$

or, since $A = \frac{1}{2}$ for small $\beta$, according as

$$\frac{\omega_*^2 d^4}{5\nu^2} > \text{ or } < F^{-2}, \tag{61b}$$

in which $\omega_* d^2/\nu$ is the Reynolds number based on $\omega_*$.

The result obtained is that

$$\mu_1 = -i\alpha^2 A_1 + O(\alpha^3).$$

For long waves, $\alpha \ll 1$, and the criterion (61) is valid. As $\alpha$ increases toward the order of 1, more terms are needed. But as long as $\alpha \ll 1$, the stability or instability found is for any period of time, however long. We were concerned with the secular instability of the flow, and we have solved the problem explicitly for long waves.

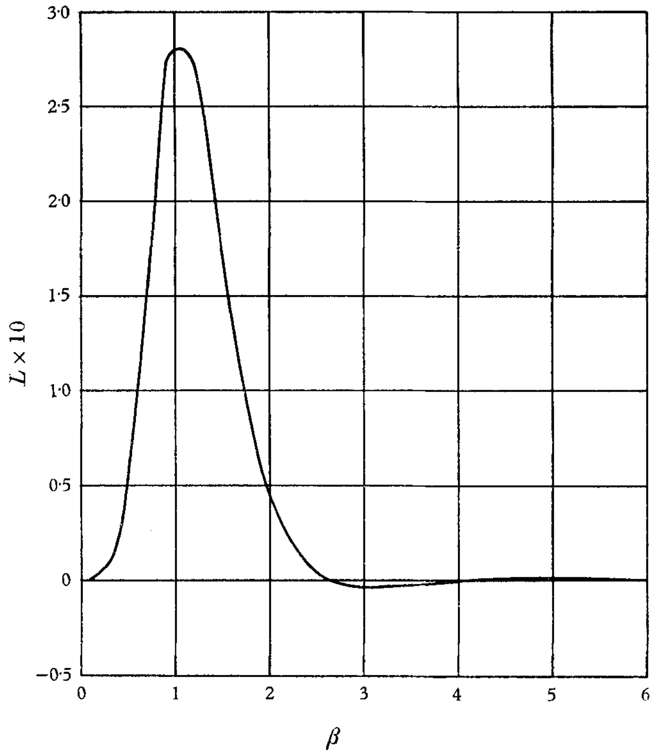


FIGURE 2. The variation of the left-hand side ($L$) of (61) with $\beta$. If $L > F^{-2}$, there is instability.

## 7. Discussion

For
$$\omega = \omega_* d/V \ll 1 \quad \text{and} \quad \beta^2 \ll 1,$$
the results obtained by the use of the approach of quasi-steadiness should be in agreement with the present results. But the former can be obtained from the results of Benjamin (1957) for the stability of the flow of a liquid layer down an inclined plane with angle of inclination $\theta$, provided we make the correspondences

$$g' = (g^2+a^2)^{\frac{1}{2}}, \quad \theta = \arctan(a/g), \tag{62}$$

in which $a$ is the (dimensional) acceleration of the plane

$$a(t) = -\omega_* V \sin \omega_* t, \tag{63}$$

and $g'$ is the $g$ in the Benjamin–Yih problem. The average velocity (dimensional) of the flow in that problem is

$$\bar{u}_a = \tfrac{1}{3}(g' \sin\theta)\, d^2/\nu = ad^2/3\nu. \tag{64}$$

Then the $\sigma_i$ in the exponential factor

$$e^{\sigma_i t} = e^{\alpha_* c_i t}$$

($\alpha_* = 2\pi/\lambda$, $\lambda$ = wavelength) is, according to the results of Benjamin,

$$\sigma_i = \frac{\alpha_*^2 \bar{u}_a^2 d^2}{\nu}\left(\frac{6}{5} - \frac{g' d \cos\theta}{3\bar{u}_a^2}\right), \tag{65}$$

the term involving surface tension being negligible. If (63) and (64) are substituted into (65), we have

$$\sigma_i = \frac{\alpha_*^2 d^2}{3\nu}\left(\frac{2a^2 d^4}{5\nu^2} - gd\right)$$

$$= \frac{\alpha_*^2 d^2}{3\nu}\left(\frac{2\omega_*^2 V^2 d^4}{5\nu^2}\sin^2\omega_* t - gd\right). \tag{66}$$

If
$$\frac{\omega_*^2 d^4}{5\nu^2} > \frac{gd}{V^2}, \tag{67}$$

the mean value of $\sigma_i$ over a long period of time is positive, and the flow is unstable in the long run. But (67) is identical to (61 *b*). Hence there is agreement, as one would expect.

One other interpretation of (66), however, is that if

$$\frac{2\omega_*^2 d^4}{5\nu^2} > \frac{gd}{V^2}, \tag{68}$$

there are intervals of time during which $\sigma_1$ is positive, and one might be tempted to conclude that the flow is unstable. But we must remember that we are, when considering *linear* stability theory of an unsteady flow, not concerned with short-range instability but with long-term instability. Provided the magnitude does not increase during the periods of positive $\sigma_i$ beyond the range of validity of the linear theory, the periods of positive $\sigma_i$ are followed (and preceded) by periods of negative $\sigma_i$ if (67) is not satisfied, and in these latter periods the disturbances will be damped, so that in the long run the flow is stable unless (67) is satisfied.

Finally we shall show that the possibility of $h_0 = 0$ and $\theta_0 \neq 0$ does lead to damped modes only. In this case it is more convenient to treat (21), (22), (25) and (28) directly. If we write

$$\phi = \Phi_0 + \alpha\Phi_1 + \alpha^2\Phi_2 + \dots, \tag{69}$$

and further assume
$$\Phi_0 = e^{\theta_0 \tau} q(y), \tag{70}$$

then $q(y)$ satisfies
$$R\theta_0 q'' = q'''', \tag{71}$$

$$q(1) = 0, \quad q'(1) = 0, \tag{72}$$

$$q''(0) = 0, \quad q'''(0) - R\theta_0 q'(0) = 0. \tag{73}$$

Multiplying (71) by $q$, integrating between zero and 1 (by parts if necessary), and utilizing the boundary conditions (72) and (73), we find that

$$R\theta_0 \int_0^1 (q')^2 dy = -\int_0^1 (q'')^2 dy,$$

so that $\theta_0$ is negative. In fact, $\theta_0$ can be found explicitly. For the solution is

$$q = A' + B' y + C' e^{\beta' y} + D' e^{-\beta' y} \quad \text{with} \quad \beta' = (R\theta_0)^{\frac{1}{2}}.$$

The boundary conditions (73) demand†

$$C' = -D', \quad B' = 0.$$

† See Yih (1963, p. 334) for a similar discussion. One evaluation for $B$ there was wrong, but the conclusions were correct.

The boundary conditions (72) then give

$$A' + B' + 2C' \sinh \beta' = 0, \quad B' + 2C'\beta' \cosh \beta' = 0,$$

the second of which gives $$\beta' \cosh \beta' = 0.$$

The solution $\beta' = 0$ must be discarded, for it leads to $q = 0$. The allowable solutions for $\beta'$ are given by

$$\beta' = \frac{2n+1}{2} \pi i,$$

or

$$R\theta_0 = -\frac{\overline{2n+1}^2}{2} \pi^2,$$

$n$ being integers. Hence all modes are damped.

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